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The Geometry of the Gauss Map

The definition of the Gauss map ; Fundamental Properties

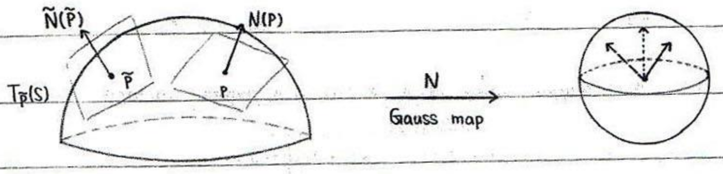
S : regular surface (orientable) , $N : S \to S^2$, N is called the Gauss map , N is C^∞

The differential of N at point p of S , $dN_p : T_p(S) \to T_{N(p)}(S^2)$ ※ $T_p(S) \,//\, T_{N(p)}(S^2)$

$\downarrow$

measure the rate of change at p of a unit normal vector field N on nbd of p

By translation, we can identify these two planes and think of $dN_p : T_p(S) \to T_p(S)$



平面上的 curve, tangent

Ex : (1) Plane , $ax(t) + by(t) + c(t) + d = 0 \to ax' + by' + cz' = 0$, $\alpha'(t) = (x', y', z') \Rightarrow N_p = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} = \text{const.}$, $dN_p = 0$

(2) hyperbolic $f(x,y) = z = y^2 - x^2$, take a parametrzation $\vec{X}(u,v) = (u, v, v^2 - u^2)$

We need to study Gauss map N near the point $p = (0,0,0)$

$X : U \subseteq \mathbb{R}^2 \to S$, $N : S \to S^2$, $N(p) = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(2u, -2v, 1)}{\sqrt{4u^2+4v^2+1}} = \frac{(u, -v, \frac{1}{2})}{\sqrt{u^2+v^2+\frac{1}{4}}}$

$\vec{X_u} = (1, 0, -2u)$, $\vec{X_v} = (0, 1, 2v) \Rightarrow \vec{X_u} \times \vec{X_v} = (2u)i + (-2v)j + (1)k$

$N(P) = N|_{p=(0,0,0)} = (0,0,1)$, $\vec{X_u}(P) = (1,0,0)$, $\vec{X_v}(P) = (0,1,0)$, $T_p(S) =$ XY plane

on the other hand , take a curve $\alpha(t) = x(u(t), v(t))$, $\alpha : I \to \mathbb{R}^3$, $\alpha(t) = (u, v, v^2 - u^2)$

$\alpha(0) = P = (0,0,0) = (u(0), v(0), v^2(0) - u^2(0))$, $u(0) = 0$, $v(0) = 0$

$\alpha'(t) = (u'(t), v'(t), 2v(t)v'(t) - 2u(t)u'(t))$, $\alpha'(0) = (u'(0), v'(0), 0) \in T_p(S)$, $T_p(S) =$ XY plane

Now, $N : S \to S^2$, $N(t) = N \circ \alpha(t)$ is a curve on S^2 , $N(t) = N(u(t), v(t)) = \frac{(u, -v, \frac{1}{2})}{\sqrt{u^2+v^2+\frac{1}{4}}}$

$\frac{dN(t)}{dt}\Big|_{t=0} = \frac{d}{dt}(N \circ \alpha(t))\Big|_{t=0} = dN_p(\alpha'(0))$

$N'(t)\big|_{t=0} = \frac{\sqrt{u^2+v^2+\frac{1}{4}}\,(u', -v', 0) - (u, -v, \frac{1}{2})\left(\sqrt{u^2+v^2+\frac{1}{4}}\right)^{\frac{1}{2}}(2uu' + 2vv')}{u^2+v^2+\frac{1}{4}}\Bigg|_{t=0} = (2u', -2v', 0)$

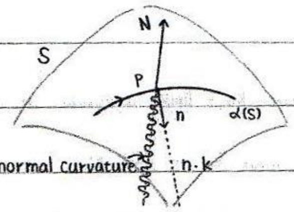
$\therefore dN_p(\alpha'(0)) = (2u'(0), -2v'(0), 0)$, $dN_p : T_p(S) \to T_{N(p)}(S^2) = T_p(S^2)$

dN_p has e-value 2 , -2 and e-vector (1,0,0) , (0,1,0) respectively

$dN_p(av+bw) = a\,dN_p(v) + b\,N_p(w)$ $\therefore dN_p$ is a self-adjoint linear map

Def. The quadratic form II_p, defined on $T_p(S)$ by $II_p(\underset{\text{vector}}{v}) = -\langle dN_p(v), v\rangle$ is called the 2nd fundamental form of S at p.

The Geometric meaning of the 2nd ff



Def. Let α be a regular curve on S passing through $p \in S$. Let k be the curvature of α at p.

N " S

$\cos\theta = \langle n, N\rangle$, where $n \equiv$ the unit normal vector of α at p.

The number $k_n = k\cos\theta = k\langle n, N\rangle$ is called the normal curvature of $\alpha \subseteq S$ at p.

$k_n \equiv$ the length of the projection of the vector $k \cdot n$ over the normal N of S at p.

Now, if α is a regular curve parametrization by arc-length and $|\alpha'(s)| = 1$

$\langle \alpha'(s), N\rangle = 0$, $\frac{d}{ds}\langle \alpha'(s), N\rangle = 0 \Rightarrow \langle \alpha''(s), N\rangle = -\langle \alpha'(s), N'(s)\rangle$

$\because II_p(\alpha'(0)) \equiv -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle = -\langle \alpha'(0), dN_p(\alpha'(0))\rangle = \langle \alpha''(0), N(p)\rangle = \langle kn, N\rangle = k\langle n, N\rangle = k_n$

k_n depends on p, v, S (如果用 β, 算出的 $II_p(\beta'(0))$ 和 α 的一樣)

Thm. (Meusnier) All curves in S with the same tangent vector must have the same normal curvature. * $\because$ 2nd ff same

Fact: If $v \in T_p(S)$ with $|v| = 1$, $II_p(v) \equiv$ the normal curvature of any curve $\alpha(t)$ passing through P with velocity v.

eigenvalue

Def. The maximum normal curvature k_1 and the minimum curvature k_2 are called the principle curvature at p.

The corresponding directions e_1, e_2 are called the principle directions.

曲線的 tangent vector = e_1 or e_2, 則曲線是?

Def. A connected regular curve $\alpha(t)$ is such that for all points $p \in \alpha$, the tangent of the curve $\alpha(t)$ is a principle direction at p, then the curve $\alpha(t)$ is called a line of curvature.

Prop. (Olinde Rodrigues) A necessary and sufficient condition for a connected curve α on S to be a line of curvature is $N'(t) = \lambda(t)\alpha'(t)$ for any parametrization curve $\alpha(t)$, where $N(t) = N(\alpha(t)) = (N\circ\alpha)(t)$, $\lambda(t)$ is differentiable function of t.

<pf> $N'(t) = \frac{d}{dt}(N\circ\alpha)(t) = dN(\alpha(t))$, $N'(t) = \lambda(t)\alpha'(t) \Longleftrightarrow -dN_{\alpha(t)}(\alpha'(t)) = -\lambda(t)\alpha'(t)$

$\Longleftrightarrow -\lambda(t)$ is an eigenvalue of $dN(\alpha(t))$, $\alpha'(t)$ is a principle direction.

Take $v \in T_p(S)$, span$\{e_1, e_2\} = T_p(S)$, $v = ae_1 + be_2$, span$\{e_1, e_2\} = T_p(S)$

Let $\{e_1, e_2\}$ be the principle directions with principle curvature k_1, k_2 respectively.

$\mathrm{II}_p(v) = -\langle dN_p(v), v\rangle = -\langle dN_p(ae_1+be_2), ae_1+be_2\rangle$

$= a^2(-\langle dN_p(e_1), e_1\rangle) + b^2(-\langle dN_p(e_2), e_2\rangle) + ab(-\langle dN_p(e_1), e_2\rangle) + ba(-\langle dN_p(e_2), e_1\rangle)$

$= k_1a^2 + k_2b^2$, where $v = ae_1 + be_2$, $\forall\, v \in T_p(S)$, $|v| = 1$

$v \equiv \cos\theta\, e_1 + \sin\theta\, e_2$, $|v| = 1$, $\mathrm{II}_p(v) = \cos^2\theta\, k_1 + \sin^2\theta\, k_2$ Euler Formula

$A = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $tr(A) = A + D$, $det(A) = AD - BC$ are invarient, if we change basis.

$N'(0) = dN_p(\alpha'(0)) = dN_p(u'(0)e_1 + v'(0)e_2) = u'(0)\,dN_p(e_1) + v'(0)\,dN_p(e_2)$

LHS of $*$, $(-u'(0), -v'(0), 0) = -u'(0)e_1 - v'(0)e_2$

$\therefore -u'(0)e_1 = u'(0)\,dN_p(e_1)$ $\quad \therefore dN_p(e_1) = -e_1 \cdots **$ $\quad -v'(0)e_2 = v'(0)\,dN_p(e_2)$ $\quad \therefore dN_p(e_2) = -e_2 \cdots **$

$\therefore$ eigenvalues $-1, -1$, eigenvectors $(1,0,0), (0,1,0)$

$\therefore k > 0$, p is called elliptic point.

Prop. Let S be a connected surface, and every point of S is an umbilical point, then S is a piece of plane or sphere.

$*$

<pf> Take a $p \in S$ and a parametrization $X : U \subseteq \mathbb{R}^2 \to S$ around P.

May assume $X(U) = V$ is a connected set, let $(u, v) \in U$

Take tangent vectors X_u, X_v which span $T_p(S)$, P is an umbilical point.

$\cdots *$

$\Rightarrow -dN_p(w) = \lambda(p)\,w, \ \forall w \in T_p(S) \quad (k_1 = k_2 = \lambda)$

Take $w = aX_u + bX_v \Rightarrow -dN_p(w) = -dN_p(aX_u + bX_v) = -a\,dN_p(X_u) - b\,dN_p(X_v) = -aN_u - bN_v$ $\cdots (1)$

On the other hand, $-dN_p(w) = -dN_p(aX_u + bX_v) \overset{*}{=} a\lambda X_u + b\lambda X_v \cdots (2)$

If $(a,b) = (1,0)$, look $(1), (2)$ $\quad \therefore -N_u = \lambda X_u \cdots (3)$; If $(a,b) = (0,1)$, $-N_v = \lambda X_v \cdots (4)$

Take derivative v to (3), $-N_{uv} = \lambda X_{uv} \cdots (5)$; take derivative u to (4), $-N_{vu} = \lambda X_{vu} \cdots (6)$

$\because X \in C^\infty$, $X_{uv} = X_{vu}$, $N_{uv} = N_{vu}$; $(5) - (6) \Rightarrow \lambda_v X_u - \lambda_u X_v = 0$ $\quad \therefore X_u = X_v = 0$, $\lambda \equiv$ const.

$*$

Now show V is either part of a plane or sphere

case 1 : $\lambda = 0$, $dN_p(w) = 0$, $\forall w \in T_p(S)$; $dN_p \equiv 0$ on $V \Rightarrow N = N_c =$ const.

$\langle X, N \rangle_u = \langle X_u, N \rangle + \langle X, N_u \rangle = 0$, $\langle X, N \rangle_v = \langle X_v, N \rangle + \langle X, N_v \rangle = 0$

$\therefore \langle X, N \rangle = \langle X, N_c \rangle \equiv$ constant. , $\langle X - C, N_c \rangle = 0$

$\uparrow$

This equation of a plane passing through $C = X(u_0, v_0)$ and $\perp N = N_c$

case 2 : If $\lambda \neq 0$, consider $X + \frac{N}{\lambda}$, if $\lambda \neq 0$.

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Prop. Let $X: U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ be a parametrization of a regular surface S that contains no umbilical pt.

Then the parametrized curves of X are lines of curvature $\Leftrightarrow f = F = 0$

<pf> Let $\alpha(t) = X(u(t), v(t))$ curve on S, $N: S \to S^2$, $dN_p: v \to dN_p(v)$

$$dN_{\alpha(t)}(\alpha'(t)) = \lambda(t)\alpha'(t) \Leftrightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = \lambda(t)\begin{pmatrix} u' \\ v' \end{pmatrix} \Leftrightarrow \begin{cases} a_{11}u' + a_{12}v' = \lambda u' \\ a_{21}u' + a_{22}v' = \lambda v' \end{cases}$$

$$\therefore \begin{cases} \left(\frac{fF - eG}{EG - F^2}\right)u' + \left(\frac{gF - fG}{EG - F^2}\right)v' = \lambda u' \\ \left(\frac{eF - fE}{EG - F^2}\right)u' + \left(\frac{fF - gE}{EG - F^2}\right)v' = \lambda v' \end{cases} \cdots \ast \qquad \because k_1 \neq k_2 \text{ at } p.$$

$(\Leftarrow)$ By $\ast$, then $-\frac{e}{E}u' = \lambda u'$, $-\frac{g}{G}v' = \lambda v'$ coordinate curve $u' = 0$ or $v' = 0 \Rightarrow u$ const or v const

$\therefore \lambda = -\frac{e}{E}, -\frac{g}{G}$ $\therefore$ coordinate curve are lines of curvature.

$(\Rightarrow)$ $\ast$: the first $\times v'$, the second $\times u'$ $\Rightarrow (fE - eG)u'v' + (gF - fG)v'^2 = (eF - fE)u'^2 + (fF - gE)v'u'$

$\Rightarrow (eF - fE)(u')^2 + (eG - gE)(u'v') - (gF - fG)(v')^2 = 0$

$$\Rightarrow \begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$ coordinate curves are line of curvature $\Rightarrow u' = 0$ or $v' = 0$

if $u' = 0$, then $Fg - Gf = 0$; if $v' = 0$, then $Ef - eF = 0$

X_u, X_v are parallel to principle directions $\therefore \langle X_u, X_v \rangle = F = 0$

$\Rightarrow Gf = 0$, $Ef = 0$ $\because k_1 \neq k_2$ $\therefore G \neq 0, E \neq 0 \Rightarrow f = 0$

下集預告：Gauss-Bonnet Theorem

Local version of Gauss-Bonnet Theorem : $\sum_{i=1}^{3}\theta_i = \pi + \int_{\Delta} K\,dA$, where K : Gauss curvature

Theorem : S is a compact surface without boundary (S is closed), $\int_S K\,dA = 2\pi \cdot \chi(S)$

where $\chi(S) = V - E + F$, Euler characteristic number

Observe: There is no asymptotic direction of an elliptic point

$\because k > 0$, $k_1 \neq 0$, $k_2 \neq 0$ and k_1, k_2 have same sign

If $w \in T_p(S)$, $w = \cos\theta\, e_1 + \sin\theta\, e_2$, θ from e_i to w in the orientation of $T_p(S)$

$k_n = \mathrm{II}_p(w) = k_1 \cos^2\theta + k_2 \sin^2\theta$ have vanish

Def. The Dupin indicatrix at p is the set of vectors $w \in T_p(S)$ st. $\mathrm{II}_p(w) = \pm 1$

Take $w \in T_p(S)$, $\{e_1, e_2\}$ principle directions with principal curvatures given by k_1 and k_2.

If $w = xe_1 + ye_2$, $\mathrm{II}_p(w) = -\langle dN_p(w), w\rangle = x^2k_1 + y^2k_2$, $x^2k_1 + y^2k_2 = \pm 1 \cdots *$

(1) Elliptic point : $k > 0 \Leftrightarrow k_1, k_2 > 0$; (i) $k_1 > 0$, $k_2 > 0$ in $*$, $1 = k_1x^2 + k_2y^2$

(ii) $k_1 < 0$, $k_2 < 0$ in $*$, $-1 = k_1x^2 + k_2y^2$

(2) If $k_1 = k_2$ (not planar plane) umbilical point : $\frac{1}{k} = x^2 + y^2$, we have circle with radius $\frac{1}{\sqrt{k}}$

(3) $k < 0$

(4) $k = 0$, $k_1 \neq 0$, $k_2 = 0$

$k = 0$, $k_1 = 0$, $k_2 \neq 0$

The Gauss Map in Local coordinates

S: regular surface (orientable), $x : U \subseteq \mathbb{R}^2 \to S \subseteq \mathbb{R}^3$ parametrization of S at p.

$T_p(S) = \mathrm{span}\{x_u, x_v\}$, $N = \frac{x_u \times x_v}{|x_u \times x_v|}$ unit normal vector on S

$\alpha(t)$ is a curve on S, $\alpha(t) = x(u(t), v(t))$, $\alpha(0) = x(u(0), v(0)) = p$, $\alpha'(t) = x_u u' + x_v v' \in T_p(S)$

$N : S \to S^2$, $dN_p : T_p(S) \to T_p(S) = T_p(S^2)$, $dN_p(\alpha'(t)) = \frac{d(N\circ\alpha)(t)}{dt} = \frac{d}{dt} N(u(t), v(t)) = N_u u' + N_v v'$

$\therefore N_u$ and $N_v \in T_p(S)$, write $N_u = a_{11} x_u + a_{21} x_v$, $N_v = a_{12} x_u + a_{22} x_v$ "equation of Weirgarten"

(write a tangent vector in matrix using x_u, x_v as a basis)

Surface of revolution

$\alpha(t) = (\overset{\overset{\circ}{\#}}{h(t)}, 0, g(t))$ be a generating curve, $x(u,v) = (h(v)\cos u, h(v)\sin u, g(v))$, $0 < u < 2\pi$, $a < v < b$

$\alpha(t)$ is a curve parametrized by arc-length, $\alpha'(t) = (h', 0, g')$, $|\alpha'(t)| = 1$, $(h')^2 + (g')^2 = 1$

$X_u = (-\sin u \cdot h, \cos u \cdot h, 0)$, $X_v = (\cos u \cdot h', \sin u \cdot h', g')$, $X_{uu} = (-\cos u \cdot h, -\sin u \cdot h, 0)$

$X_{uv} = (-\sin u \cdot h', \cos u \cdot h', 0)$, $X_{vv} = (\cos u \cdot h'', \sin u \cdot h'', g'')$, $X_u \times X_v = (hg'\cos u, hg'\sin u, -hh')$

$E = \langle X_u, X_u \rangle = h^2$, $F = \langle X_u, X_v \rangle = 0$, $G = \langle X_v, X_v \rangle = (h')^2 + (g')^2 = 1$, $N = (g'\cos u, g'\sin u, -h')$

$e = \langle N, X_{uu} \rangle = -hg'$, $f = \langle N, X_{uv} \rangle = 0$, $g = \langle N, X_{vv} \rangle = h''g' - g''h'$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-hg'(h''g' - g''h')}{h^2} = \frac{-g'(h''g' - g''h')}{h}$$

$$H = \frac{eG - 2fF + Eg}{2(EG - F^2)} = \frac{1}{2}\left(\frac{-hg' + h^2(h''g' - g''h')}{h^2}\right) = \frac{1}{2} \cdot \frac{-g' + h(h''g' - g''h')}{h}$$

* $K = 0$ (1) $g' = 0$, $\alpha'(t) = (h', 0, 0)$

$\therefore$ The tangent vector to the generating curve α is perpendicular to z-axis.

(2) $h''g' - g''h' = 0$, $\alpha(t) = (h(t), 0, g(t))$, $\alpha'(t) = (h'(t), 0, g'(t))$, $\alpha''(t) = (h'', 0, g'')$

$\therefore$ The curvature of the generating curve $\alpha(t)$ is zero.

$k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$ curvature of α, $\alpha' \times \alpha'' = (h''g' - h'g'')\hat{j}$

(3) $g' = 0$ and $h''g' - h'g'' = 0 \Leftrightarrow$ we have a planer $(e = f = g = 0)$

principle curvature $dN \sim \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \sim \begin{pmatrix} -\frac{e}{E} & 0 \\ 0 & -\frac{g}{G} \end{pmatrix}$ $\because f = 0$, $F = 0$

$k = \begin{cases} -\frac{e}{E} = +\frac{g'}{h'} \\ \frac{g}{G} = h''g' - h'g'' \end{cases}$, $K = k_1 k_2$, $H = \frac{k_1 + k_2}{2}$, $K = \frac{-g'(h''g' - g''h')}{h} = \frac{-h''}{h}$

$(h')^2 + (g')^2 = 1$, $2hh'' + 2gg'' = 0 \Rightarrow h'h'' = -g'g''$

Ex: $x(u,v) = ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v)$, $h(v) = a + r\cos v$, $g(v) = r\sin v$

$K = \frac{-h''}{h} = \frac{r\cos v}{a + r\cos v}$ 和上週不同 $\because (h')^2 + (g')^2 = r^2$ $\therefore K = \frac{r\cos v}{a + r\cos v} \cdot \frac{1}{r^2}$

Hw: $\vec{X}(u,v) = (u, v, f(u,v))$, $f \in C^\infty \Rightarrow$ find K, H. 討論 $K > 0, = 0, < 0$

Prop. Let S be a regular orientable surface, then Gauss curvature and mean curvature are smooth function. However, the principle curvature k_1, k_2 are continuous and smooth on the open set of non-umbilical points.

<pf> $H^2 - K = 0 \iff \left(\frac{k_1+k_2}{2}\right)^2 - k_1 k_2 = 0 \iff k_1 = k_2$

Ex: Torus 甜甜圈 $x(u,v) = ((a+r\cos u)\cos v, (a+r\cos u)\sin v, r\sin u)$, $0 < u, v < 2\pi$, $0 < r < a$

Find Gauss curvature, mean curvature of the points of the torus covered by $x(u,v)$

$x_u = (-r\sin u \cdot \cos v, -r\sin u \cdot \sin v, r\cos u)$, $x_v = (-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0)$

$x_{uu} = (-r\cos u \cdot \cos v, -r\cos u \cdot \sin v, -r\sin u)$, $x_{vv} = (-(a+r\cos u)\cos v, -(a+r\cos u)\sin v, 0)$

$x_{uv} = (r\sin u \cdot \sin v, -r\sin u \cdot \cos v, 0)$, $N = \frac{x_u \times x_v}{|x_u \times x_v|} = \frac{x_u \times x_v}{\sqrt{EG - F^2}}$

$x_u \times x_v = (-r(a+r\cos u)\cos u \cdot \cos v, -r(a+r\cos u)\cos u \cdot \sin v, -r(a+r\cos u)\sin u) \cdots *$

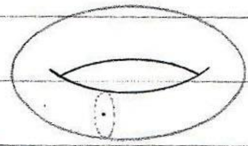
$E = \langle x_u, x_u \rangle = r^2$, $F = \langle x_u, x_v \rangle = 0$, $G = \langle x_v, x_v \rangle = (a+r\cos u)^2$

$\Rightarrow |x_u \times x_v| = \sqrt{EG - F^2} = r(a+r\cos u)$

$e = \langle N, x_{uu} \rangle = \frac{1}{r(a+r\cos u)} \langle *, x_{uu} \rangle = r$, $f = \langle N, x_{uv} \rangle = 0$

$g = \langle N, x_{vv} \rangle = \frac{1}{r(a+r\cos u)} \langle *, x_{vv} \rangle = (a+r\cos u)\cos u$

$K = \frac{eg - f^2}{EG - F^2} = \frac{\cos u}{r(a+r\cos u)}$, $H = \frac{eG - 2fF + Eg}{2(EG - F^2)} = \frac{1}{2}\left(\frac{(a+r\cos u) + r\cos u}{r(a+r\cos u)}\right)$



$K(x(u,v)) = \sim$ the point on Torus covered by $x(u,v)$

$K = 0$, if $u = \frac{\pi}{2}, \frac{3\pi}{2}$ two line on the torus

$K > 0$, if $0 < u < \frac{\pi}{2}$, $\frac{3\pi}{2} < u < 2\pi$ outer the torus, all points in this region are elliptic point

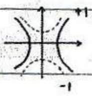
$K < 0$, if $\frac{\pi}{2} < u < \frac{3\pi}{2}$ inner the torus, all points in this region are hyperbolic point case 2 : p is a hyperbolic point, $k(p) < 0$, $\exists$ principle directions e_1 and e_2

st. $\mathrm{II}_p(e_1) > 0$ and $\mathrm{II}_p(e_2) < 0 \Rightarrow$ given $\varepsilon > 0$, $\mathrm{II}_p(\varepsilon e_1) > 0$ and $\mathrm{II}_p(\varepsilon e_2) < 0$

$\therefore$ S must lie on both sides of tangent plane.

Dupin indicatrix : $\mathrm{II}_p(v) = \pm 1$, $\forall v \in T_p(S)$; $\forall (x, y) \in S$ st. $k_1 x^2 + k_2 y^2 = \pm 1$, $k_i \equiv$ normal curvature, $i = 1, 2$

observe		Dupin indicatrix
Elliptic point	$K > 0$	$k_1 > 0$, $k_2 > 0$ $\quad k_1 x^2 + k_2 y^2 = 1$ $k_1 < 0$, $k_2 < 0$ $\quad k_1 x^2 + k_2 y^2 = -1$
Hyperbolic point	$K < 0$	$k_1 > 0$, $k_2 < 0$ $\quad k_1 x^2 + k_2 y^2 = \pm 1$
Umbilical point	$k_1 = k_2$, $dN \neq 0$	$k_1 = k_2 = k$, $x^2 + y^2 = \frac{1}{k^2}$, circle with radius $\frac{1}{k}$
Parabolical point	$dN \neq 0$	$k_1 = 0$, $k_2 < 0$ or $k_1 > 0$, $k_2 = 0$ parallel lines



Ex : (1) $X(u, v) = (u, v, u^3 - 3v^2 u)$, $k(p) = 0$, Monkey saddle

$X_u = (1, 0, 3u^2 - 3v^2)$, $X_v = (0, 1, -6vu)$, $X_{uu} = (0, 0, 6u)$, $X_{uv} = (0, 0, -6v)$, $X_{vv} = (0, 0, -6u)$

$N = \frac{X_u \times X_v}{|X_u \times X_v|}$, $X_u \times X_v = (3v^2 - 3u^2, 6uv, 1)$, $EG - F^2 = |X_u \times X_v|^2$

$E = \langle X_u, X_u \rangle = 1 + (3u^2 - 3v^2)^2$, $F = \langle X_u, X_v \rangle = -18uv(u^2 - v^2)$, $G = \langle X_v, X_v \rangle = 1 + 36u^2v^2$

$e = \langle N, X_{uu} \rangle = \frac{6u}{|X_u \times X_v|}$, $f = \langle N, X_{uv} \rangle = \frac{-6v}{|X_u \times X_v|}$, $g = \langle N, X_{vv} \rangle = \frac{-6u}{|X_u \times X_v|}$

$K = \frac{eg - f^2}{|X_u \times X_v|} = \frac{-36u^2 - 36v^2}{[1 + (3u^2 - 3v^2)]^2 (1 + 36u^2v^2) - 18u^2v^2(u^2 - v^2)^2}$, $k(p) = 0 \Leftrightarrow u^2 = -v^2$

$p = 0$, $u = v = 0$, p is a planer point, $\forall x \in$ Nbd V of p, they lie on both sides of $T_p(S)$.

(2) $z = y^3$, $dN \neq 0$, $k = 0$ at parabolic point

(3) 圓柱 $k = 0$, $\forall x \in$ Nbd V of p lies on the same side of $T_p(S)$

Note. The three examples shows No such Theorem for parabolic point and planer point.

No. 12

Date 106 : 12 : 19

Ex : helicoid 麻花卷 $x(u,v) = (v\cos u,\ v\sin u,\ \lambda u)$, $\lambda \in \mathbb{R} \setminus \{0\}$

$x_u = (-v\sin u,\ v\cos u,\ \lambda)$, $x_v = (\cos u,\ \sin u,\ 0)$, $x_{uu} = (-v\cos u,\ -v\sin u,\ 0)$, $x_{vv} = (0,0,0)$

$x_{uv} = (-\sin u,\ \cos u,\ 0)$, $E = \langle x_u, x_u \rangle = v^2 + \lambda^2$, $F = \langle x_u, x_v \rangle = 0$, $G = \langle x_v, x_v \rangle = 1$

$e = \langle N, x_{uu} \rangle = \frac{1}{\sqrt{EG-F^2}} \begin{vmatrix} x_u \\ x_v \\ x_{uv} \end{vmatrix} = 0$, $f = \langle N, x_{uv} \rangle = \frac{\lambda}{\sqrt{\lambda^2+v^2}}$, $g = \langle N, x_{vv} \rangle = 0$

$K = \frac{eg-f^2}{EG-F^2} = \frac{-\lambda^2}{(\lambda^2+v^2)^2} < 0$, $H = \frac{eG-2fF+Eg}{2(EG-F^2)} = 0$ $\therefore$ helicoid is complete covered by hyper. pts

Hw : elliptic paraboloid $S = \{(x,y,z) \mid 2z = x^2+y^2\}$ Find $K(x(u,v))$, $H(x(u,v))$

Prop. Let $p \in S$ be an elliptic point, then exists nbd V of p in S st. for all points in V, they must belong to the same side of the tangent plane at p.

Let $p \in S$ be a hyperbolic point, then in each nbd N of p, there exists points of S in both sides of $T_p(S)$.

<pf> Take a parametrization at p, $x : U \to S \subseteq \mathbb{R}^3$, $p = x(0,0)$, $q = x(u,v) \in S$

Define $d(q) \equiv$ signed distance function $q \in S$ to tangent plane $T_p(S)$

Think of $d : U \to \mathbb{R}$ st. $d(u,v) = d(x(u,v))$

Write $d(u,v) = \langle x(u,v) - x(0,0), N \rangle$, where $N \equiv$ unit normal vector of S

$\because x(u,v) \in C^\infty$ $\therefore$ write $d(u,v)$ in Taylor's expansion around $p = (0,0)$

Look $x(u,v) = x(0,0) + x_u(0,0)u + x_v(0,0)v + \frac{1}{2}\left[x_{uu}(0,0)u^2 + 2x_{uv}(0,0)uv + x_{vv}(0,0)v^2\right] + R$

Hence, $d(u,v) = N \cdot (x(u,v) - x(0,0)) = N(p) \cdot [x_u(0,0)u + x_v(0,0)v + \quad]$

$N \perp x_u$, $N \perp x_v$

$d(u,v)$ has the same sign $(eu^2 + 2fuv + gv^2)$

If $w = ux_u + vx_v$ and $\lim_{|h| \to 0} \frac{\langle R, N \rangle}{|w|^2} \to 0$, $\mathrm{II}_p(w) = eu^2 + 2fuv + gv^2 = k_n$

case I : p is an elliptic point, $k(p) > 0$, i.e. $e_1, e_2 \Rightarrow \mathrm{II}_p(e_1) \overset{<0}{>0}$ and $\mathrm{II}_p(e_2) \overset{<0}{>0}$ or

For the points close to p, they must be on the one side of $T_p(S)$.

Let $\vec{x}(u,v)$ be a parametrization at $p \in S$ with $x(0,0) = P$, let $e(u,v) = e$, $f(u,v) = f$, $g(u,v) = g$

If a regular connected curve $\alpha(t)$ in the coordinate neighborhood of X is an asymptotic curve

$\Leftrightarrow$ for any parametrization $\alpha(t) = X(u(t), v(t))$, $t \in I$, $\mathrm{II}_p(\alpha') = 0$, $e(u')^2 + 2fu'v' + g(v') = 0$ --- $*$

$*$ is called the differential equation of the asymptotic curve.

Prop. Let S be an oriention regular surface and P be a hyperbolic point on S.

If X is parametrization at $p \in S$ ($X: U \subseteq \mathbb{R}^2 \to V \subset S$) the coordinate curve of the parametrization are asymptotic curves $\Leftrightarrow$ $e = g = 0$

<pf> ($\Rightarrow$) $\because$ P is hyperbolic point, $K = \frac{eg - f^2}{EG - F^2}$, $eg - f^2 < 0$

Let $\alpha(t) = X(u_0, v(t))$, $\beta(t) = X(u(t), v_0)$ be two coordinate curves of X.

$\alpha' = X_u \overset{0}{u'} + X_v v' = X_v v'$, $\beta' = X_u u' + X_v \overset{0}{v'} = X_u u'$

By $*$, $\mathrm{II}_p(\alpha') = g(v')^2 = 0$, $\mathrm{II}_p(\beta') = e(u')^2 = 0$ $\Rightarrow$ $e = 0$, $g = 0$

($\Leftarrow$) $e = 0$, $g = 0$ $\Rightarrow$ from $*$, $2fu'v' = 0$; $\alpha(t) = X(u(t), v(t))$ $\Rightarrow$ $u' = 0$ or $v' = 0$ $\therefore$ $\alpha(t) = X(u(t), v_0)$ or $\alpha(t) = X(u_0, v(t))$

Ex: Prove that the asymptotic curve on the surface $X(u,v) = (u\cos v, u\sin v, \ln u)$ are given by $\ln u = \pm(v + c)$, $C \equiv$ constant, $u \neq 0$

<pf> $X_u = (\cos v, \sin v, \frac{1}{u})$, $X_v = (-u\sin v, u\cos v, 0)$, $E = 1 + \frac{1}{u^2}$, $F = 0$, $G = u^2$, $\sqrt{EG - F^2} = \sqrt{1 + u^2}$

$X_{uu} = (0, 0, -\frac{1}{u^2})$, $X_{uv} = (-\sin v, \cos v, 0)$, $X_{vv} = (-u\cos v, -u\sin v, 0)$, $N = \frac{(-\cos v, -\sin v, u)}{\sqrt{1+u^2}}$

$e = \frac{1}{\sqrt{1+u^2}}(-\frac{1}{u})$, $f = 0$, $g = \frac{u}{\sqrt{1+u^2}}$ $\because$ $\alpha(t)$ is an asymptotic curve $\therefore$ $e(u')^2 + 2fu'v' + g(v')^2 = 0$

$\frac{1}{\sqrt{1+u^2}}(-\frac{1}{u})(u')^2 + 0 + \frac{u}{\sqrt{1+u^2}}(v')^2 = 0$ $\Rightarrow$ $\frac{1}{u\sqrt{1+u^2}}(-(u')^2 + (uv')^2) = 0$

$\therefore$ $\frac{(u')^2}{(v')^2} = u^2$ $\Rightarrow$ $\frac{du}{dv} = \pm u$ $\therefore$ $\ln u = \pm(v + c)$

No. 10

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$dN(\alpha') = N_u u' + N_v v' = (a_{11}x_u + a_{21}x_v)u' + (a_{12}x_u + a_{22}x_v)v' = (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v$

$II_p(\alpha') = -\langle dN_p(\alpha'), \alpha' \rangle = -\langle (a_{11}u' + a_{12}v')x_u + (a_{21}u' + a_{22}v')x_v,\ x_u u' + x_v v' \rangle$

On the other hand, $II_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle = -\langle N_u u' + N_v v', x_u u' + x_v v' \rangle$

$= -\langle N_u, x_u \rangle (u')^2 - \langle N_v, x_v \rangle (v')^2 - \langle N_u, x_v \rangle u'v' - \langle N_v, x_u \rangle u'v'$

$\because$ dN is self-adjoint, $\langle N_u, x_v \rangle = \langle N_v, x_u \rangle$

$\because \langle x_u, N \rangle = 0$, $\langle x_u, N \rangle_u = 0 \Leftrightarrow \langle x_{uu}, N \rangle + \langle x_u, N_u \rangle = 0 \Rightarrow -\langle N_u, x_u \rangle = \langle x_{uu}, N \rangle = e$

$\langle x_u, N \rangle_v = 0 \Leftrightarrow -\langle N_v, x_u \rangle = \langle x_{uv}, N \rangle = f$, $\langle x_v, N \rangle = 0 \Leftrightarrow -\langle N_v, x_v \rangle = \langle x_{vv}, N \rangle = g$

where e, f, g are the coefficient of 2nd f.f

$\therefore II_p(\alpha') = e(u')^2 + 2f u'v' + g(v')^2$ ··· *

$e = \langle N, x_{uu} \rangle = -\langle N_u, x_u \rangle = -\langle a_{11}x_u + a_{21}x_v, x_u \rangle = -a_{11}E - a_{21}F$ ··· (1)

$f = \langle N, x_{uv} \rangle = -\langle N_u, x_v \rangle = -\langle a_{11}x_u + a_{21}x_v, x_v \rangle = -a_{11}F - a_{21}G$ ··· (2)

$= \langle N, x_{vu} \rangle = -\langle N_v, x_u \rangle = -\langle a_{12}x_u + a_{22}x_v, x_u \rangle = -a_{12}E - a_{22}F$ ··· (3)

$g = \langle N, x_{vv} \rangle = -\langle N_v, x_v \rangle = -\langle a_{12}x_u + a_{22}x_v, x_v \rangle = -a_{12}F - a_{22}G$ ··· (4)

From (1) ~ (4), we have $-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \frac{1}{EG-F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$

$K = \det A = a_{11}a_{22} - a_{12}a_{21} = \frac{eg - f^2}{EG - F^2}$

Next, find the formula for principal curvature k_1, k_2 and mean curvature

$\det\begin{pmatrix} a_{11}+k & a_{12} \\ a_{21} & a_{22}+k \end{pmatrix} = 0 \Rightarrow (a_{11}+k)(a_{22}+k) - a_{12}a_{21} = 0 \Rightarrow k^2 + (a_{11}+a_{22})k + (a_{11}a_{22} - a_{12}a_{21}) = 0$ ··· **

$\because k_1$ and k_2 are solution of ** $\therefore -k_1$ and $-k_2$ are eigenvalues of dN

$\det(dN + kI) = \det\left(\begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\right) = k^2 - (k_1+k_2)k + k_1k_2$ ··· ***

From ** and ***, $H = -\frac{1}{2}(a_{11} + a_{22}) = \frac{eG - 2fF + Eg}{2(EG-F^2)}$, $K = a_{11}a_{22} - a_{12}a_{21} = \frac{eg - f^2}{EG - F^2}$

From ***, $k^2 - 2Hk + K = 0$, $k_1, k_2 = H \pm \sqrt{H^2 - K}$ ··· ****

$(X+\frac{N}{\lambda})_u = (X_u + \frac{\lambda N_u}{\lambda^2}) = 0$, $(X+\frac{N}{\lambda})_v = (X_v + \frac{\lambda N_v}{\lambda^2}) = 0$ $\because N_u = -\lambda X_u,\ N_v = -\lambda X_v$

$X+\frac{N}{\lambda} \equiv$ constant vector $= X_0$, $|X - X_0|^2 = \frac{1}{\lambda^2}$ ⋯ * $\because |N| = 1$

This show a sphere centered at X_0 with radius $\frac{1}{|\lambda|}$

$\therefore$ all points in V st. * holds $\therefore V$ is a piece of sphere.

* Next, we want to show the global vector.

Take $\alpha(t)$ on the surface, $\alpha : [0,1] \to S$

By previous argument, $\exists$ an open nbd. $V(t)$ of $\alpha(t)$, $V(t)$ is a piece of plane or sphere.

consider $\alpha^{-1}(V(t))$ open covering of $[0,1]$. $\because [0,1]$ is a closed interval.

By Heine-Borel, $\exists$ a finite t_i st. $\bigcup_{i=1}^{m} \alpha^{-1}(V(t_i))$ cover $[0,1]$

$\Rightarrow \bigcup_{i=1}^{m} V(t_i)$ covers the curve (β) in S. $\therefore \beta$ has to be on the same plane or sphere at p.

Since β is arbitrary on $S \Rightarrow S$ is a piece of a plane or a sphere.

Note. If S is not connected, we can't have such result.

漸近方向 $\mathrm{II}_p(v) = k_n = 0$

Def. Let $p \in S$ an asymptotic direction of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An asymptotic curve of S is a regular connected curve $C \subseteq S$ st. for all $p \in C$, the tangent line is given by an asymptotic direction.

* Ex: A straight line on a surface is an asymptotic curve.

Given a straight line parametrized by arc-length $\alpha(t) = p + tv$, where $|v| = 1$

$\alpha'(t) = v$, $\alpha''(t) = 0$, normal curvature $k_n = k\langle n, N\rangle$, $\langle \alpha''(t), N\rangle = k\langle n, N\rangle = k_n = 0$

* If $\alpha(t)$ is a curve with positive curvature and $\alpha(t)$ is an asymptotic curve.

$\Leftrightarrow$ its binormal vector b of $\alpha(t)$ is parallel to N, where $N \equiv$ unit normal vector of S.

Def. $\det(-dN_p) = \det(dN_p)$ = Gauss curvature of S at p, i.e. $K = k_1 k_2$

mean curvature $H = \frac{1}{2}\operatorname{tr}(-dN_p) = \frac{k_1+k_2}{2}$, N = unit normal vector on S

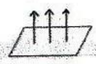
However, if we change the orientation of the surface, the determinant does not change, but the trace changes sign.

Def. A point $p \in S$ is called :

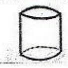
(1) elliptic point , if $\det(dN_p) > 0$, $K > 0$ $\quad S^2(r)$, $k_1 = k_2 = \frac{1}{r}$

(2) hyperbolic point , if $K < 0$ $\quad x(u,v) = (u, v, v^2 - u^2)$, $k_1 = 2$, $k_2 = -2$

(3) planer , if $K = 0$ & $dN = 0$



(4) parabolic point , if $K = 0$, but $dN \neq 0$



$k_1 = 1$, $k_2 = 0$

(5) umbilical point , if $k_1 = k_2$ $\quad S^2(r)$, plane (planer points are umbilical pt.)

Ex : $S = \{ (x,y,z) \in \mathbb{R}^3 \mid 2z = x^2 + y^2 \}$

regular surface : (1) $f(x,y,z) = x^2 + y^2 - 2z \in C^\infty$, 0 is a regular value of f, $f^{-1}(0) = S$.

(2) $\left(u, v, \frac{u^2+v^2}{2}\right)$ graph of g

study the Gauss map near point $p = (0,0,0)$ on S , $u(0) = 0$, $v(0) = 0$

$X(u,v) = \left(u, v, \frac{u^2+v^2}{2}\right)$, $X_u = (1, 0, u)$, $X_v = (0, 1, v)$ $\Rightarrow$ $X_u(p) = (1,0,0)$, $X_v(p) = (0,1,0)$

$N = \frac{X_u \times X_v}{|X_u \times X_v|} = \frac{(-u,-v,1)}{\sqrt{u^2+v^2+1}}$, $N(p) = (0,0,1)$, where $X_u \times X_v = (-u)\,i + (-v)\,j + k$

$\alpha : I \to S$, $N : S \to S^2$ $\therefore N(t) = (N \circ \alpha)t$ is a curve on S^2

$\alpha(t) = x(u(t), v(t)) = \left(u(t), v(t), \frac{u^2(t) + v^2(t)}{2}\right)$, $\alpha(0) = P = (0,0,0) = \left(u(0), v(0), \frac{u^2(0) + v^2(0)}{2}\right)$

$\alpha'(t) = (u'(t), v'(t), u(t)v'(t) + u'(t)v(t))$, $\alpha'(0) = (u'(0), v'(0), 0) \in T_p(S) \equiv X-Y$ plane

$N(t) = N(\alpha(t)) = \frac{(-u,-v,1)}{\sqrt{u^2+v^2+1}}$, $N'(t)\big|_{t=0} = \dfrac{\sqrt{u^2+v^2+1}\,(-u',-v',0) - (-u,-v,1)\frac{uu'+vv'}{\sqrt{u^2+v^2+1}}}{u^2+v^2+1} = (-u'(0), -v'(0), 0)$ --- *

$\alpha'(0) = (u'(0), v'(0), 0) = u'(0)e_1 + v'(0)e_2$

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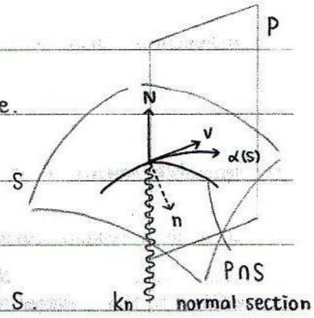
Ex : $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, $v = (x', y', z')$, $2xx' + 2yy' + 2zz' = 0 \Rightarrow N = (-x, -y, -z)$ or $\bar{N} = (x, y, z)$

Fix N orientation , $II_p(v) = -\langle dN_p(v), v\rangle = -\langle -v, v\rangle = 1 = k_n$, $k_n = k\langle n, N\rangle = 1 \cdot \cos 0° = 1$

Fix $\bar{N}$ orientation , $II_p(v) = -\langle d\bar{N}_p(v), v\rangle = -\langle v, v\rangle = -1 = k_n$, $k_n = k\cos\theta = (-1)(-1) = 1$



Observation that $k_n = II_p(v)$ depends on the orientation of the surface.

$P \equiv$ plane = spanned by t and N

Let $\alpha(S)$ be a parametrized curve and $t = v = \alpha'(0)$ tangent vector at p.

Take the plane P determined by t and N , $N \equiv$ unit normal vector of S.

$P \cap S \equiv$ new curve which is called normal section.

Since the normal section lies on S and P , its normal vector is a vector of P and is perpendicular to $t = v$

Hence , it must be N or $-N$ $\therefore k_n = k\langle n, \underset{N \text{ or } -N}{N}\rangle = \pm k$

$|k_n| \equiv$ curvature of the normal section of S at p along $\alpha'(0) = t$ direction.

$II_p : T_p(S) \to \mathbb{R}$, $\forall\, v \in T_p(S)$ $\because dN_p$ is linear and self-adjoint $\therefore dN_p$ can be diagonalized

$\exists\ \lambda_1$ and λ_2 two eigenvalues corresponding eigenvectors e_1, e_2 st. $dN_p(e_1) = \lambda_1 e_1$, $dN_p(e_2) = \lambda_2 e_2$

$\therefore dN_p = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, where $\{e_1, e_2\}$ forms a basis of $T_p(S)$.

convention : May assume $k_1 \geq k_2$, $k_1 = \max\limits_{\substack{v \in T_p(S) \\ |v| = 1}} -\langle dN_p(v), v\rangle = \max\limits_{\substack{v \in T_p(S) \\ |v| = 1}} II_p(v)$

maximum normal curvature at p

$k_2 = \min\limits_{\substack{v \in T_p(S) \\ |v| = 1}} -\langle dN_p(v), v\rangle = \min\limits_{\substack{v \in T_p(S) \\ |v| = 1}} II_p(v)$

minimum normal curvature at p

Ex : $N : S^2 \to S^2$, $S^2 = \{ (x, y, z) \mid x^2+y^2+z^2 = 1 \}$; $\alpha(t) = (x(t), y(t), z(t))$, $\alpha : I \to \mathbb{R}^3$ regular curve

$2xx' + 2yy' + 2zz' = 0$, $\alpha'(t) = (x'(t), y'(t), z'(t)) \in T_p(S^2)$

$\therefore N = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = (x, y, z)$, $dN_p(\alpha'(0)) = (x', y', z') = \alpha'(0) = v$, $dN_p(v) = v$, $\forall v \in T_p(S^2)$

fixed positive orientation , $N(x, y, z) = \frac{(-x, -y, -z)}{\sqrt{x^2+y^2+z^2}} = (-x, -y, -z)$, $dN_p(\alpha'(0)) = (-x', -y', -z') = -\alpha'(0) = -v$

Ex : cylinder $X(U) = \{ (x, y, z) \mid x^2+y^2 = 1 \}$ 先 check regular surface , use prop. 2

$f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2+y^2$, 1 is a regular value of f $\Rightarrow f^{-1}(1)$ is a regular surface.

$\alpha(t) = (x(t), y(t), z(t))$, $\alpha(0) = p$, $\alpha'(t) = (x'(t), y'(t), z'(t))$, $2xx' + 2yy' = 0$

$\tilde{N} = (x, y, 0)$, $N = (-x, -y, 0)$ (if we fix positive orientation)

case 1 : $v \in T_p(S)$, $v = (0, 0, A)$, $A \neq 0 \Rightarrow dN_p(v) = 0\,v$ (its behavior like plane)

case 2 : $w \in T_p(S)$, w is parallel to XY plane $\Rightarrow dN_p(w) = -w$

$\therefore 0, -1$ are eigenvalues of dN_p

Hw : $z = x^2 + \lambda y^2$, λ is a positive number , find dN_p and its eigenvalues

Def. A diff. map $A : V \to V$ is self-adjoint linear map , if $\langle A(v), w \rangle = \langle v, A(w) \rangle$, $\forall v, w \in V$

Prop. The diff map $dN_p : T_p(S) \to T_{N(p)}(S^2) = T_p(S)$ of Gauss map is a self-adjoint linear map.

<pf> We need to show $\langle dN_p(v), w \rangle = \langle v, dN_p(w) \rangle$, $\forall v, w \in T_p(S)$

If $\vec{X} : U \to S$ is a parametrization of S , hope $\langle dN_p(X_u), X_v \rangle = \langle X_u, dN_p(X_v) \rangle$

Let we define $N_u = dN_p(X_u)$, $N_v = dN_p(X_v)$, N is normal vector on S , $N \perp X_u$, $N \perp X_v$

$\langle N, X_u \rangle = 0$, $\langle N, X_v \rangle = 0$, we have $\langle N_v, X_u \rangle = -\langle N, X_{uv} \rangle$, $\langle N_u, X_v \rangle = -\langle N, X_{vu} \rangle$

$\because \vec{X} \in C^\infty$, $X_{uv} = X_{vu}$ $\therefore \langle N_v, X_u \rangle = \langle N_u, X_v \rangle = \langle dN_p(X_v), X_u \rangle = \langle X_v, dN_p(X_u) \rangle$.

$\langle u, v \rangle = \overline{\langle v, u \rangle}$